VECTOR-VALUED WALSH–PALEY MARTINGALES AND GEOMETRY OF BANACH SPACES

BY

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ABSTRACT

The concept of Rademacher type p ($1 \leq p \leq 2$) plays an important role in the local theory of Banach spaces. In [3] Mascioni considers a weakening of this concept and shows that for a Banach space X weak Rademacher type p implies Rademacher type r for all $r < p$.

As with Rademacher type p and weak Rademacher type p, we introduce the concept of Haar type p and weak Haar type p by replacing the Rademacher functions by the Haar functions in the respective definitions. We show that weak Haar type p implies Haar type r for all $r < p$. This solves a problem left open by Pisier [5].

The method is to compare Haar type ideal norms related to different index sets.

1. Introduction

Let (r_n) denote the sequence of Rademacher functions. For $1 \leq p \leq 2$, we say that a Banach space X is of **Rademacher type** p if there exists a constant $c \geq 0$ such that

$$\left\| \sum_{k=1}^{n} x_k r_k \middle| L_p \right\| \leq c \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$.

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Furthermore, for $1 \le p < 2$, we say that X is of **weak Rademacher type** p if there exists a constant $c \ge 0$ such that

$$\left\| \sum_{k=1}^{n} x_k r_k \middle| L_2 \right\| \le c\, n^{1/p-1/2} \left(\sum_{k=1}^{n} \|x_k\|^2 \right)^{1/2}$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$.

Kahane's Inequality and the Cauchy–Schwarz Inequality tell us that Rademacher type p implies weak Rademacher type p. Mascioni [3] has shown a partial converse.

THEOREM 1.1 (Mascioni [3]): *If a Banach space X is of weak Rademacher type p for some $1 \le p < 2$ then it is of Rademacher type r for all $r < p$.*

A different concept can be introduced by using sequences of X-valued martingale differences instead of Rademacher sequences. Following Pisier [6], we say that a Banach space X is of **martingale type** p ($1 \le p \le 2$) if there exists a constant $c \ge 0$ such that

$$\left\| \sum_{k=1}^{n} d_k \middle| L_p \right\| \le c \left(\sum_{k=1}^{n} \|d_k | L_p\|^p \right)^{1/p}$$

for all $n \in \mathbb{N}$ and all X-valued martingale difference sequences $d_1, \ldots, d_n$.

Again, for $1 \le p < 2$, we say that X is of **weak martingale type** p if there exists a constant $c \ge 0$ such that

$$\left\| \sum_{k=1}^{n} d_k \middle| L_2 \right\| \le c\, n^{1/p-1/2} \left(\sum_{k=1}^{n} \|d_k | L_2\|^2 \right)^{1/2}$$

for all $n \in \mathbb{N}$ and all X-valued martingale difference sequences $d_1, \ldots, d_n$.

In [5] Pisier showed a theorem similar to Theorem 1.1.

THEOREM 1.2 (Pisier [5]): *If a Banach space X is of weak martingale type p for some $1 \le p < 2$ then it is of martingale type r for all $r < p$.*

Denoting by $\chi_k^{(j)}$ the Haar functions, we see that the sequence

$$\left(\sum_{j=1}^{2^{k-1}} x_k^{(j)} \chi_k^{(j)} \right), \quad k = 1, \ldots, n, \quad (x_k^{(j)}) \subseteq X$$

forms a sequence of X-valued martingale differences. Restricting to those martingales (usually referred to as dyadic or Walsh–Paley martingales) in the definition of martingale type, we can define the apparently weaker concept of Haar type. For $1 \le p \le 2$, we say that a Banach space X is of **Haar type** p if there exists a constant $c \ge 0$ such that

$$\Big\| \sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} x_k^{(j)} \chi_k^{(j)} \Big| L_p \Big\| \le c \Big(\sum_{k=1}^{n} \Big\| \sum_{j=1}^{2^{k-1}} x_k^{(j)} \chi_k^{(j)} \Big| L_p \Big\|^p \Big)^{1/p}$$

for all $n \in \mathbb{N}$ and $(x_k^{(j)}) \subseteq X$.

Once more it was Pisier [5] who showed that Haar type p and martingale type p coincide.

THEOREM 1.3 (Pisier [5]): *A Banach space X is of martingale type p if and only if it is of Haar type p.*

Of course, the next step is to define weak Haar type. For $1 \le p < 2$, we say that a Banach space X is of **weak Haar type** p if there exists a constant $c \ge 0$ such that

$$\Big\| \sum_{k=1}^{n} \sum_{j=1}^{2^{k-1}} x_k^{(j)} \chi_k^{(j)} \Big| L_2 \Big\| \le c\, n^{1/p-1/2} \Big(\sum_{k=1}^{n} \Big\| \sum_{j=1}^{2^{k-1}} x_k^{(j)} \chi_k^{(j)} \Big| L_2 \Big\|^2 \Big)^{1/2}$$

for all $n \in \mathbb{N}$ and $(x_k^{(j)}) \subseteq X$.

The main result of this paper is the following companion to Theorems 1.1 and 1.2.

THEOREM 1.4: *If a Banach space X is of weak Haar type p for some $1 \le p < 2$ then it is of Haar type r for all $r < p$.*

This solves a problem in [5], where Pisier could show the same conclusion under the stronger assumption that X is of weak martingale type p.

Let us now quickly review the contents of this article section by section.

In Section 2, we define the necessary concepts. In Section 3, we formulate and prove the main new ingredient of the proof of the main theorem. In Section 4 we provide a lemma of combinatorial character needed to prove a local variant of the main theorem. In Section 5, we show how to derive the main theorem from the results in Section 3. Finally, in Section 6, we consider some examples and formulate some problems.

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2. Definitions

For $k = 1, 2, \ldots$ and $j = 0, \pm 1, \pm 2, \ldots$, we define the **Haar functions** by

$$\chi_k^{(j)}(t) := \begin{cases} +2^{(k-1)/2} & \text{if } t \in \Delta_k^{(2j-1)}, \\ -2^{(k-1)/2} & \text{if } t \in \Delta_k^{(2j)}, \\ 0 & \text{otherwise.} \end{cases}$$

Here

$$\Delta_k^{(j)} := \left[\tfrac{j-1}{2^k}, \tfrac{j}{2^k}\right)$$

are the **dyadic intervals.**

The following facts are obvious consequences of the definition of the Haar functions:

$$\chi_k^{(j)}(t - \tfrac{1}{2^{k-1}}) = \chi_k^{(j+1)}(t) \quad \text{and} \tag{1}$$

$$\chi_{k+1}^{(j)}(t) = \sqrt{2}\chi_k^{(j)}(2t). \tag{2}$$

We denote by

$$\mathbb{D} := \{(k, j) \colon k = 1, 2, \ldots; j = 1, \ldots, 2^{k-1}\}$$

the **dyadic tree**. For a finite subset $\mathbb{F} \subseteq \mathbb{D}$, we consider the orthonormal system

$$\mathcal{H}(\mathbb{F}) := \{\chi_k^{(j)} \colon (k, j) \in \mathbb{F}\} \subseteq L_2[0, 1).$$

In particular, we will consider finite dyadic trees

$$\mathbb{D}_m^n := \{(k, j) \colon k = m, \ldots, n; j = 1, \ldots, 2^{k-1}\},$$

where $m \le n$. For $t \in [0, 1)$, we also use the branches

$$\mathbb{B}(t) := \{(k, j) \colon t \in \Delta_{k-1}^{(j)}\} = \{(k, j) \colon \chi_k^{(j)}(t) \neq 0\}.$$

These are exactly those points of the dyadic tree $\mathbb{D}$ which lie on one of the infinite paths starting in the root $(1, 1)$. Figure 1 shows a part of the set $\mathbb{D}$, where the thicker dots stand for the first elements in any of the sets $\mathbb{B}(t)$ for $t \in [\frac{3}{16}, \frac{1}{4})$.

We define the following type ideal norms associated with the systems $\mathcal{H}(\mathbb{F})$.

Definition: For $T \in \mathfrak{L}(X, Y)$ and for a finite set $\mathbb{F} \subseteq \mathbb{D}$ denote by $\tau(T|\mathcal{H}(\mathbb{F}))$ the least constant $c \ge 0$ such that

$$\left\| \sum_{\mathbb{F}} T x_k^{(j)} \chi_k^{(j)} \middle| L_2 \right\| \le c \left(\sum_{\mathbb{F}} \|x_k^{(j)}\|^2 \right)^{1/2} \tag{3}$$

for all $\{x_k^{(j)}: (k,j) \in \mathbb{F}\} \subseteq X$.

We call the map

$$\tau(\mathcal{H}(\mathbb{F})): T \longrightarrow \tau(T|\mathcal{H}(\mathbb{F}))$$

the **Haar type ideal norm** associated with the index set $\mathbb{F}$.

We write $\tau(X|\mathcal{H}(\mathbb{F}))$ instead of $\tau(I_X|\mathcal{H}(\mathbb{F}))$, where I_X is the identity map of the Banach space X.

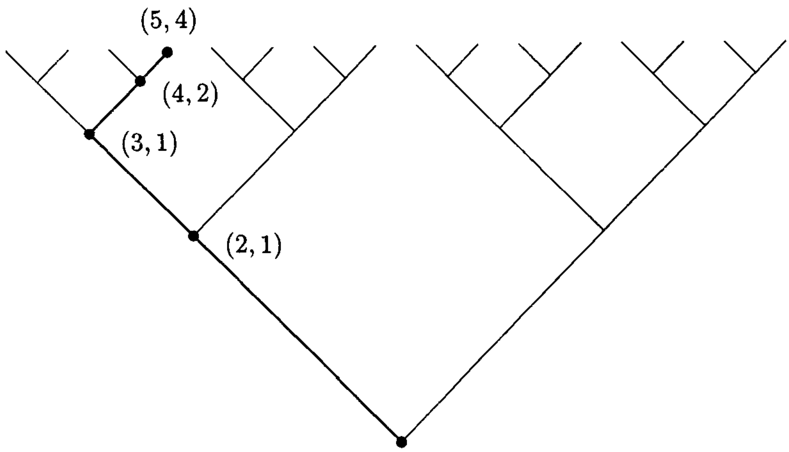

Figure 1. The dyadic tree $\mathbb{D}$.

3. Comparison of Haar type ideal norms

In this section we compare Haar type ideal norms associated with different index sets $\mathbb{F}$.

The following proposition is an easy consequence of the facts (1) and (2).

PROPOSITION 3.1: *Let $n \geq m \geq 1$. Then for all $T \in \mathfrak{L}(X,Y)$*

$$\tau(T|\mathcal{H}(\mathbb{D}_{m+1}^{n+1})) \leq \tau(T|\mathcal{H}(\mathbb{D}_m^n)).$$

Proof: We split $\mathbb{D}_{m+1}^{n+1}$ into the two subtrees

$$\mathbb{S}_1 := \left\{(k,j) \in \mathbb{D}_{m+1}^{n+1}: 1 \leq j \leq 2^{k-2}\right\},$$
$$\mathbb{S}_2 := \left\{(k,j) \in \mathbb{D}_{m+1}^{n+1}: 2^{k-2}+1 \leq j \leq 2^{k-1}\right\}.$$

Note that for $0 \leq t < 1/2$

$$(k,j) \in \mathbb{S}_2 \quad \text{implies} \quad \chi_k^{(j)}(t) = 0$$

and for $1/2 \le t < 1$

$$(k,j) \in \mathbb{S}_1 \quad \text{implies} \quad \chi_k^{(j)}(t) = 0.$$

Hence we may write

$$\begin{aligned}
&\int_0^1 \Big\| \sum_{\mathbb{D}_{m+1}^{n+1}} Tx_k^{(j)} \chi_k^{(j)}(t) \Big\|^2 dt \\
&= \int_0^{1/2} \Big\| \sum_{\mathbb{D}_{m+1}^{n+1}} Tx_k^{(j)} \chi_k^{(j)}(t) \Big\|^2 dt + \int_{1/2}^1 \Big\| \sum_{\mathbb{D}_{m+1}^{n+1}} Tx_k^{(j)} \chi_k^{(j)}(t) \Big\|^2 dt \\
&= \int_0^{1/2} \Big\| \sum_{\mathbb{S}_1} Tx_k^{(j)} \chi_k^{(j)}(t) \Big\|^2 dt + \int_{1/2}^1 \Big\| \sum_{\mathbb{S}_2} Tx_k^{(j)} \chi_k^{(j)}(t) \Big\|^2 dt.
\end{aligned}$$

Substituting $s = 2t$ and $h = k - 1$ and taking into account formula (2), the first integral can be estimated as follows:

$$\begin{aligned}
\int_0^{1/2} \Big\| \sum_{\mathbb{S}_1} Tx_k^{(j)} \chi_k^{(j)}(t) \Big\|^2 dt &= \frac{1}{2} \int_0^1 \Big\| \sum_{h=m}^{n} \sum_{j=1}^{2^{h-1}} Tx_{h+1}^{(j)} \chi_{h+1}^{(j)} \left(\frac{s}{2}\right) \Big\|^2 ds \\
&\le \tau(T|\mathcal{H}(\mathbb{D}_m^n))^2 \sum_{h=m}^{n} \sum_{j=1}^{2^{h-1}} \|x_{h+1}^{(j)}\|^2 \\
&= \tau(T|\mathcal{H}(\mathbb{D}_m^n))^2 \sum_{\mathbb{S}_1} \|x_k^{(j)}\|^2.
\end{aligned}$$

Replacing t by $t + \frac{1}{2}$ and j by $j + 2^{k-2}$, and using (1), the estimate above passes into

$$\int_{1/2}^1 \Big\| \sum_{\mathbb{S}_2} Tx_k^{(j)} \chi_k^{(j)}(t) \Big\|^2 dt \le \tau(T|\mathcal{H}(\mathbb{D}_m^n))^2 \sum_{\mathbb{S}_2} \|x_k^{(j)}\|^2,$$

which implies the desired inequality. ∎

By iterated application of Proposition 3.1, we obtain the following result.

Corollary 3.2: *Let $m, n \in \mathbb{N}$. Then we have for all $T \in \mathfrak{L}(X, Y)$ that*

$$\tau(T|\mathcal{H}(\mathbb{D}_{m+1}^{m+n})) \le \tau(T|\mathcal{H}(\mathbb{D}_1^n)).$$

The following transformation and their effects on the Haar functions will play the crucial role in the proof of Proposition 3.7. For $(h,i) \in \mathbb{D}$, we denote by $\varphi_h^{(i)}$ the transformation of $[0,1)$ that interchanges the intervals

$$\Delta_{h+1}^{(4i-2)} \quad \text{and} \quad \Delta_{h+1}^{(4i-1)}.$$

More formally

$$\varphi_h^{(i)}(t) := \begin{cases} t + \frac{1}{2^{h+1}} & \text{for } t \in \Delta_{h+1}^{(4i-2)}, \\ t - \frac{1}{2^{h+1}} & \text{for } t \in \Delta_{h+1}^{(4i-1)}, \\ t & \text{otherwise.} \end{cases} \tag{4}$$

Lemma 3.3 describes the behavior of the Haar functions under $\varphi_h^{(i)}$. What is $\chi_k^{(j)} \circ \varphi_h^{(i)}$ for $(k,j) \in \mathbb{D}$? It turns out that the most interesting cases are those if (k,j) belongs to the fork $\mathbb{F}_h^{(i)} := \{(h,i),(h+1,2i-1),(h+1,2i)\}$.

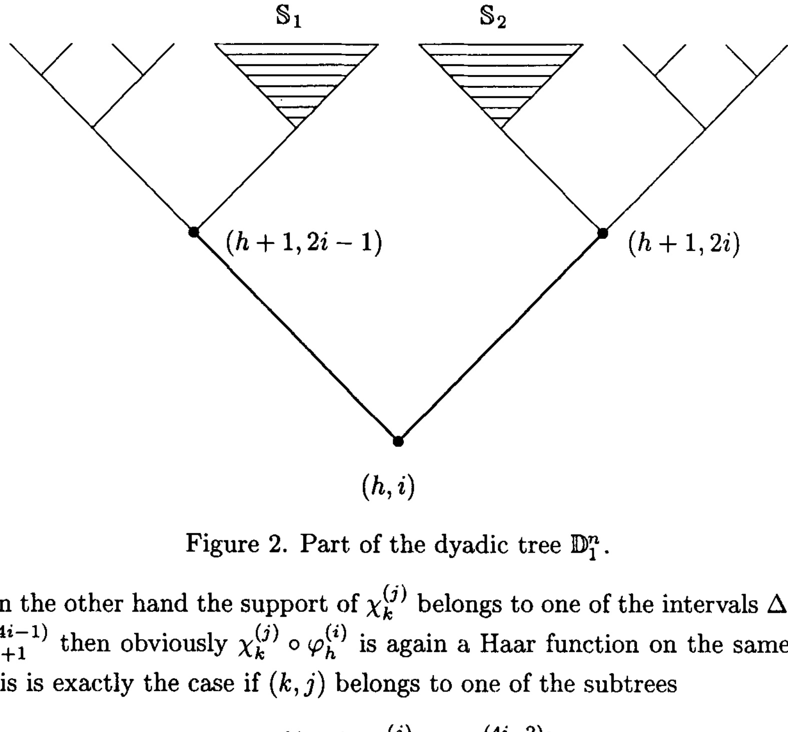

Figure 2. Part of the dyadic tree $\mathbb{D}_1^n$.

If on the other hand the support of $\chi_k^{(j)}$ belongs to one of the intervals $\Delta_{h+1}^{(4i-2)}$ or $\Delta_{h+1}^{(4i-1)}$ then obviously $\chi_k^{(j)} \circ \varphi_h^{(i)}$ is again a Haar function on the same level k. This is exactly the case if (k,j) belongs to one of the subtrees

$$\mathbb{S}_1 := \{(k,j)\colon \Delta_{k-1}^{(j)} \subseteq \Delta_{h+1}^{(4i-2)}\},$$
$$\mathbb{S}_2 := \{(k,j)\colon \Delta_{k-1}^{(j)} \subseteq \Delta_{h+1}^{(4i-1)}\}.$$

In the remaining cases the Haar functions $\chi_k^{(j)}$ are invariant under $\varphi_h^{(i)}$. Look at

Figure 2 to see how the different index sets $\mathbb{F}_h^{(i)}$, $\mathbb{S}_1$ and $\mathbb{S}_2$ are related to each other.

LEMMA 3.3: *On the fork $\mathbb{F}_h^{(i)}$ the transformation $\varphi_h^{(i)}$ acts as follows:*

$$\begin{aligned} \chi_h^{(i)} \circ \varphi_h^{(i)} &= \big(\chi_{h+1}^{(2i-1)} + \chi_{h+1}^{(2i)}\big)/\sqrt{2}, \\ \chi_{h+1}^{(2i-1)} \circ \varphi_h^{(i)} &= \big(\sqrt{2}\chi_h^{(i)} + \chi_{h+1}^{(2i-1)} - \chi_{h+1}^{(2i)}\big)/2, \\ \chi_{h+1}^{(2i)} \circ \varphi_h^{(i)} &= \big(\sqrt{2}\chi_h^{(i)} - \chi_{h+1}^{(2i-1)} + \chi_{h+1}^{(2i)}\big)/2. \end{aligned}$$

Moreover, for $(k,j) \in \mathbb{S}_1$ or $(k,j) \in \mathbb{S}_2$, we get

$$\chi_k^{(j)} \circ \varphi_h^{(i)} = \begin{cases} \chi_k^{(j+2^{k-h-2})} & \text{if } (k,j) \in \mathbb{S}_1, \\ \chi_k^{(j-2^{k-h-2})} & \text{if } (k,j) \in \mathbb{S}_2. \end{cases}$$

The remaining Haar functions $\chi_k^{(j)}$ are invariant under $\varphi_h^{(i)}$.

Proof: Looking at Figure 3, we easily see that

$$\begin{aligned} \chi_h^{(i)} \circ \varphi_h^{(i)} &= \big(\chi_{h+1}^{(2i-1)} + \chi_{h+1}^{(2i)}\big)/\sqrt{2}, \\ \big(\chi_{h+1}^{(2i-1)} + \chi_{h+1}^{(2i)}\big) \circ \varphi_h^{(i)} &= \sqrt{2}\chi_h^{(i)}, \\ \big(\chi_{h+1}^{(2i-1)} - \chi_{h+1}^{(2i)}\big) \circ \varphi_h^{(i)} &= \chi_{h+1}^{(2i-1)} - \chi_{h+1}^{(2i)}. \end{aligned}$$

Solving this system of equations, one gets the assertions for $(k,j) \in \mathbb{F}_h^{(i)}$. The other assertions follow easily from the definition of $\varphi_h^{(i)}$. ∎

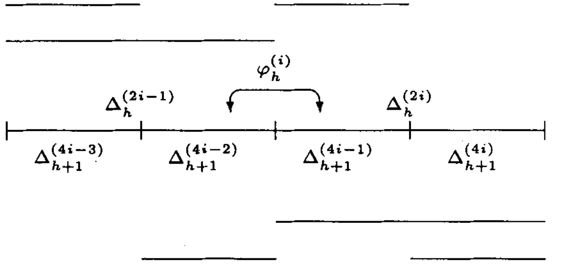

Figure 3. The graphs of the functions $\chi_h^{(i)}$, $\chi_{h+1}^{(2i-1)}$ and $\chi_{h+1}^{(2i)}$.

The following concept turns out to be very useful in the proof of the main theorem.

Definition: Let $\mathbb{F} \subseteq \mathbb{D}$ be a finite subset. The **local height** of $\mathbb{F}$ is defined as the maximal number of indices in $\mathbb{F}$, that are contained in one branch $\mathbb{B}(t)$. That is, we let

$$\mathrm{lh}(\mathbb{F}) := \max_{t \in [0,1)} |\mathbb{F} \cap \mathbb{B}(t)|.$$

The next definition is due to the special behavior of the Haar functions $\chi_k^{(j)}$ under $\varphi_h^{(i)}$ for the indices (k,j) in the fork $\mathbb{F}_h^{(i)}$.

Definition: For given $\mathbb{F} \subseteq \mathbb{D}$, we say that an index $(h,i) \in \mathbb{F}$ is **$\mathbb{F}$-admissible**, if its successors $(h+1, 2i-1)$ and $(h+1, 2i)$ do not belong to $\mathbb{F}$.

For an $\mathbb{F}$-admissible index $(h,i) \in \mathbb{F}$, we assign to $\mathbb{F}$ another index set $\Phi_h^{(i)}(\mathbb{F})$ as follows. We let

$$(h+1, 2i-1), (h+1, 2i) \in \Phi_h^{(i)}(\mathbb{F}).$$

Moreover, we let $(k, j^*) \in \Phi_h^{(i)}(\mathbb{F})$ for all $(k,j) \in \mathbb{F} \setminus \mathbb{F}_h^{(i)}$, where j^* is determined by

$$\chi_k^{(j)} \circ \varphi_h^{(i)} = \chi_k^{(j^*)}.$$

Note that by Lemma 3.3 such a number j^* exists.

The transformations $\Phi_h^{(i)}$ obviously enjoy the following properties.

LEMMA 3.4: *Let $(h,i) \in \mathbb{F}$ be $\mathbb{F}$-admissible. Then*

$$|\Phi_h^{(i)}(\mathbb{F})| = |\mathbb{F}| + 1 \quad \textit{and} \quad \mathrm{lh}(\Phi_h^{(i)}(\mathbb{F})) = \mathrm{lh}(\mathbb{F}).$$

The next lemma clarifies the importance of the index set $\Phi_h^{(i)}(\mathbb{F})$.

LEMMA 3.5: *Let $(h,i) \in \mathbb{F}$ be $\mathbb{F}$-admissible. If $f_{\mathbb{F}}$ is given by*

$$f_{\mathbb{F}} = \sum_{\mathbb{F}} x_k^{(j)} \chi_k^{(j)} \quad \textit{with } x_k^{(j)} \in X,$$

then there exist elements $y_k^{(j)} \in X$ such that

$$f_{\mathbb{F}} \circ \varphi_h^{(i)} = \sum_{\Phi_h^{(i)}(\mathbb{F})} y_k^{(j)} \chi_k^{(j)}.$$

Moreover, the l_2-sums of the families $(x_k^{(j)})$ and $(y_k^{(j)})$ are the same,

$$\sum_{\mathbb{F}} \|x_k^{(j)}\|^2 = \sum_{\Phi_h^{(i)}(\mathbb{F})} \|y_k^{(j)}\|^2.$$

Proof: Write $f_{\mathbb{F}}$ in the form

$$f_{\mathbb{F}} = x_h^{(i)}\chi_h^{(i)} + \sum_{\mathbb{F}_0} x_k^{(j)}\chi_k^{(j)}$$

with $\mathbb{F}_0 = \mathbb{F} \setminus \{(h,i)\}$. In view of Lemma 3.3

$$f_{\mathbb{F}} \circ \varphi_h^{(i)} = \tfrac{1}{\sqrt{2}} x_h^{(i)} \chi_{h+1}^{(2i-1)} + \tfrac{1}{\sqrt{2}} x_h^{(i)} \chi_{h+1}^{(2i)} + \sum_{\mathbb{F}_0} x_k^{(j)} \chi_k^{(j^*)}.$$

Again, j^* is determined by

$$\chi_k^{(j)} \circ \varphi_h^{(i)} = \chi_k^{(j^*)}.$$

Note that

$$\sum_{\mathbb{F}_0} x_k^{(j)} \chi_k^{(j^*)} = \sum_{\Phi_h^{(i)}(\mathbb{F}_0)} x_k^{(j^*)} \chi_k^{(j)}.$$

Hence the elements $y_k^{(j)}$ are given by

$$y_{h+1}^{(2i-1)} = \tfrac{1}{\sqrt{2}} x_h^{(i)}, \; y_{h+1}^{(2i)} = \tfrac{1}{\sqrt{2}} x_h^{(i)} \quad \text{and}$$
$$y_k^{(j)} = x_k^{(j^*)} \quad \text{for } (k,j) \in \Phi_h^{(i)}(\mathbb{F}_0),$$

and their l_2-sum can easily be computed. ∎

Note that an index set $\mathbb{F}$ of local height n can be arbitrarily large. However, the following lemma shows that, using the transforms $\Phi_h^{(i)}$, such an index set can be compressed so as to fit into a certain dyadic tree $\mathbb{D}_{m+1}^{m+n}$. The idea is to replace an admissible index $(h,i) \in \mathbb{F}$ by its successors $(h+1, 2i-1)$ and $(h+1, 2i)$.

LEMMA 3.6: *Let $n := \mathrm{lh}(\mathbb{F})$. Then there exist $m \in \mathbb{N}$ and a finite sequence of transforms*

$$\mathbb{F} =: \mathbb{F}_0 \xrightarrow{\Phi_{h_1}^{(i_1)}} \mathbb{F}_1 \xrightarrow{\Phi_{h_2}^{(i_2)}} \dots \xrightarrow{\Phi_{h_N}^{(i_N)}} \mathbb{F}_N$$

such that

$$\mathbb{F}_N \subseteq \mathbb{D}_{m+1}^{m+n}.$$

Proof: Since $\mathbb{F}$ is finite, we can choose $m \in \mathbb{N}$ with $\mathbb{F} \subseteq \mathbb{D}_1^{m+n}$. As often as possible, we successively apply transforms $\Phi_{h_l}^{(i_l)}$ for $\mathbb{F}_{l-1}$-admissible indices (h_l, i_l)

$$\mathbb{F}_{l-1} \xrightarrow{\Phi_{h_l}^{(i_l)}} \mathbb{F}_l.$$

In order to guarantee that $\mathbb{F}_l$ is still contained in $\mathbb{D}_{m+1}^{m+n}$, we always assume that $h_l < m+n$.

Since

$$|\mathbb{F}_l| = |\mathbb{F}_{l-1}| + 1 \qquad \text{and} \qquad |\mathbb{F}_l| \le |\mathbb{D}_1^{m+n}|,$$

this process terminates after a finite number of steps. Hence, the last index set $\mathbb{F}_N$ does not contain any $\mathbb{F}_N$-admissible index (h,i) with $h < m+n$.

Choose h such that

$$\mathbb{F}_N \subseteq \mathbb{D}_h^{m+n} \qquad \text{and} \qquad \mathbb{F}_N \not\subseteq \mathbb{D}_{h+1}^{m+n}.$$

If $h = m+n$, then $\mathbb{F}_N \subseteq \mathbb{D}_{m+n}^{m+n} \subseteq \mathbb{D}_{m+1}^{m+n}$. Thus we are done. Otherwise, take any index $(h, j_0) \in \mathbb{F}_N$ on the lowest level. Since $h < m+n$ and (h, j_0) is not $\mathbb{F}_N$-admissible, at least one of its successors, say $(h+1, j_1)$, must belong to $\mathbb{F}_N$. In this way, we find a sequence

$$(h, j_0),\ (h+1, j_1), \ldots, (m+n, j_{m+n-h}) \in \mathbb{F}_N$$

of length $m+n-h+1$, that belongs to some branch $\mathbb{B}(t)$. Hence

$$m+n-h+1 \le n = \mathrm{lh}(\mathbb{F}).$$

Finally, we conclude from $m+1 \le h$ that

$$\mathbb{F}_N \subseteq \mathbb{D}_h^{m+n} \subseteq \mathbb{D}_{m+1}^{m+n}. \qquad \blacksquare$$

We can now formulate the most interesting result of this section.

PROPOSITION 3.7: *Let $\mathbb{F} \subseteq \mathbb{D}$ be a finite subset of local height n. Then there exists $m \in \mathbb{N}$ such that for all $T \in \mathfrak{L}(X,Y)$*

$$\tau(T|\mathcal{H}(\mathbb{F})) \le \tau(T|\mathcal{H}(\mathbb{D}_{m+1}^{m+n})).$$

Proof: Consider the transforms $\Phi_{h_1}^{(i_1)}, \ldots, \Phi_{h_N}^{(i_N)}$ constructed in the previous Lemma. Let $\varphi_{h_1}^{(i_1)}, \ldots, \varphi_{h_N}^{(i_N)}$ denote the associated bijections defined in (4).

Given

$$f_{\mathbb{F}} = \sum_{\mathbb{F}} x_k^{(j)} \chi_k^{(j)},$$

we let

$$f_{\mathbb{F}_l} := f_{\mathbb{F}_{l-1}} \circ \varphi_{h_l}^{(i_l)} \qquad \text{and} \qquad f_{\mathbb{F}_0} := f_{\mathbb{F}}.$$

Write

$$f_{\mathbb{F}_l} = \sum_{\mathbb{F}_l} x_k^{(j,l)} \chi_k^{(j)}.$$

Since the transformations $\varphi_h^{(i)}$ are measure preserving, we have

$$\left\| \sum_{\mathbb{F}} Tx_k^{(j)} \chi_k^{(j)} \middle| L_2 \right\| = \|f_{\mathbb{F}_0}|L_2\| = \cdots = \|f_{\mathbb{F}_N}|L_2\| = \left\| \sum_{\mathbb{F}_N} Tx_k^{(j,N)} \chi_k^{(j)} \middle| L_2 \right\|.$$

Moreover, Lemma 3.5 yields that

$$\sum_{\mathbb{F}} \|x_k^{(j)}\|^2 = \sum_{\mathbb{F}_0} \|x_k^{(j,0)}\|^2 = \cdots = \sum_{\mathbb{F}_N} \|x_k^{(j,N)}\|^2.$$

Since $\mathbb{F}_N \subseteq \mathbb{D}_{m+1}^{m+n}$, we have

$$\left\| \sum_{\mathbb{F}_N} Tx_k^{(j,N)} \chi_k^{(j)} \middle| L_2 \right\| \le \tau(T|\mathcal{H}(\mathbb{D}_{m+1}^{m+n})) \left(\sum_{\mathbb{F}_N} \|x_k^{(j,N)}\|^2 \right)^{1/2}.$$

Hence

$$\left\| \sum_{\mathbb{F}} Tx_k^{(j)} \chi_k^{(j)} \middle| L_2 \right\| \le \tau(T|\mathcal{H}(\mathbb{D}_{m+1}^{m+n})) \left(\sum_{\mathbb{F}} \|x_k^{(j)}\|^2 \right)^{1/2}. \quad \blacksquare$$

We now summarize the results of this section.

THEOREM 3.8: *Let $\mathbb{F} \subseteq \mathbb{D}$ be a finite subset of local height n. Then we have for all $T \in \mathfrak{L}(X,Y)$ that*

$$\tau(T|\mathcal{H}(\mathbb{F})) \le \tau(T|\mathcal{H}(\mathbb{D}_1^n)).$$

Proof: The assertion follows immediately from Corollary 3.2 and Proposition 3.7. ∎

Remark: Using the same methods as in the proof of Proposition 3.7, one can show that

$$\tau(T|\mathcal{H}(\mathbb{F})) = \tau(T|\mathcal{H}(\mathbb{D}_1^n))$$

for all $T \in \mathfrak{L}(X,Y)$, if $\mathbb{F}$ has **exact local height** n, i.e. if

$$|\mathbb{F} \cap \mathbb{B}(t)| = n$$

for all $t \in [0,1)$.

As a consequence of the remark above, we get the following result.

COROLLARY 3.9: *Let $m, n \in \mathbb{N}$. Then we have for all $T \in \mathfrak{L}(X, Y)$ that*

$$\tau(T|\mathcal{H}(\mathbb{D}_{m+1}^{m+n})) = \tau(T|\mathcal{H}(\mathbb{D}_1^n)).$$

4. A combinatorial lemma

In this section we provide a lemma, which is needed in the proof of Theorem 5.3.

LEMMA 4.1: *Let $\mathbb{F} \subseteq \mathbb{D}_1^n$ be an index set of local height $l \leq n$. If $|\mathbb{F}| < 2^l - 1$ then there exists an index $(k, j) \in \mathbb{D}_1^n \setminus \mathbb{F}$ such that*

$$\mathrm{lh}(\mathbb{F} \cup \{(k, j)\}) = l.$$

Proof: For $l = 1$ the assertion is trivial. Now assume that for $l - 1$ the lemma is true. To prove the lemma for l, we use induction over $n \geq l$.

If $n = l$ then $\mathbb{D}_1^n$ has exactly $2^n - 1 = 2^l - 1$ elements and each subset of $\mathbb{D}_1^n$ has local height less than or equal to l. Hence, we may take any element $(k, j) \in \mathbb{D}_1^n \setminus \mathbb{F}$. Since $|\mathbb{F}| < 2^l - 1$, the latter set is certainly nonempty.

Now assume that $\mathbb{F} \subseteq \mathbb{D}_1^n$ and $n > l$. Note that the subtrees

$$\mathbb{S}_1 := \left\{(k, j) \in \mathbb{D}_2^n : 1 \leq j \leq 2^{k-2}\right\},$$
$$\mathbb{S}_2 := \left\{(k, j) \in \mathbb{D}_2^n : 2^{k-2} + 1 \leq j \leq 2^{k-1}\right\}$$

can be canonically identified with $\mathbb{D}_1^{n-1}$. Let

$$\mathbb{F}_1 := \mathbb{F} \cap \mathbb{S}_1 \quad \text{and} \quad \mathbb{F}_2 := \mathbb{F} \cap \mathbb{S}_2$$

and consider them as subsets of $\mathbb{D}_1^{n-1}$ via the identification above.

If $(1, 1) \notin \mathbb{F}$ then

$$\mathrm{lh}(\mathbb{F}_1) \leq l \quad \text{and} \quad |\mathbb{F}_1| \leq |\mathbb{F}| < 2^l - 1.$$

By the induction hypothesis, there exists $(k, j) \in \mathbb{S}_1 \setminus \mathbb{F}_1$ such that

$$\mathrm{lh}(\mathbb{F}_1 \cup \{(k, j)\}) \leq l.$$

Since $(1, 1) \notin \mathbb{F}$, this implies that also

$$\mathrm{lh}(\mathbb{F} \cup \{(k, j)\}) \leq l.$$

If however $(1, 1) \in \mathbb{F}$ then

$$\mathrm{lh}(\mathbb{F}_1) \leq l - 1 \quad \text{and} \quad \mathrm{lh}(\mathbb{F}_2) \leq l - 1.$$

Moreover, without loss of generality, we may assume that

$$|\mathbb{F}_1| \le \frac{|\mathbb{F}-1|}{2} < 2^{l-1} - 1.$$

Since we already know that the lemma is true for $l-1$, there exists $(k,j) \in \mathbb{D}_1^{n-1} \setminus \mathbb{F}_1$ such that

$$\mathrm{lh}(\mathbb{F}_1 \cup \{(k,j)\}) \le l-1.$$

Since $(1,1) \in \mathbb{F}$, this implies that

$$\mathrm{lh}(\mathbb{F} \cup \{(k,j)\}) \le l. \qquad \blacksquare$$

By repeated application of the previous lemma, one easily checks the following statement.

COROLLARY 4.2: *Let $\mathbb{F} \subseteq \mathbb{D}_1^n$ be an index set satisfying $\mathrm{lh}(\mathbb{F}) \le l \le n$. If $|\mathbb{F}| < 2^l - 1$ then there exists an index set $\mathbb{F}_0 \subseteq \mathbb{D}_1^n \setminus \mathbb{F}$ of cardinality $2^l - 1 - |\mathbb{F}|$ such that*

$$\mathrm{lh}(\mathbb{F} \cup \mathbb{F}_0) \le l.$$

5. Haar type and weak Haar type

First of all, let us introduce variants of the ideal norms $\tau(\mathcal{H}(\mathbb{D}_1^n))$ defined in Section 2.

Definition: For $T \in \mathfrak{L}(X,Y)$ and for $1 \le p \le 2$ denote by $\tau_p(T|\mathcal{H}(\mathbb{D}_1^n))$ the least constant $c \ge 0$ such that

$$\left\| \sum_{\mathbb{D}_1^n} T x_k^{(j)} \chi_k^{(j)} \middle| L_p \right\| \le c \left(\sum_{k=1}^{n} \left\| \sum_{j=1}^{2^{k-1}} x_k^{(j)} \chi_k^{(j)} \middle| L_p \right\|^p \right)^{1/p}$$

for all $(x_k^{(j)}) \subseteq X$.

Note that

$$\tau_2(\mathcal{H}(\mathbb{D}_1^n)) = \tau(\mathcal{H}(\mathbb{D}_1^n)).$$

The following theorem allows one to easily determine the asymptotic behavior of the ideal norms $\tau_p(\mathcal{H}(\mathbb{D}_1^n))$. It is a refinement of a theorem of Pisier in [5] and is due to Geiß [2].

THEOREM 5.1: *Let $n \in \mathbb{N}$ and $1 \le p \le 2$ be fixed. For $T \in \mathfrak{L}(X,Y)$ assume that*

$$\left\| \sum_{\mathbb{D}_1^n} T x_k^{(j)} \chi_k^{(j)} \Big| L_1 \right\| \le c_n \left\| \left(\sum_{k=1}^{n} \left\| \sum_{j=1}^{2^{k-1}} x_k^{(j)} \chi_k^{(j)} \right\|^p \right)^{1/p} \Big| L_\infty \right\|$$

for all $(x_k^{(j)}) \subseteq X$. Then it follows that

$$\tau_p(T|\mathcal{H}(\mathbb{D}_1^n)) \le c\, c_n.$$

Here c is a universal constant not depending on n nor p.

The next definitions are motivated by the corresponding concepts in the case of Rademacher functions.

Definition: For $1 \le p \le 2$, we say that an operator $T \in \mathfrak{L}(X,Y)$ is of **Haar type** p, if there exists a constant $c \ge 0$ such that

$$\tau_p(T|\mathcal{H}(\mathbb{D}_1^n)) \le c$$

for all $n \in \mathbb{N}$.

For $1 \le p < 2$, we say that an operator $T \in \mathfrak{L}(X,Y)$ is of **weak Haar type** p, if there exists a constant $c \ge 0$ such that

$$\tau(T|\mathcal{H}(\mathbb{D}_1^n)) \le c\, n^{1/p-1/2}$$

for all $n \in \mathbb{N}$.

Remarks:

- Theorem 5.1 ensures that for $p < 2$ Haar type p implies weak Haar type p. Indeed, note that

$$\left\| \sum_{\mathbb{D}_1^n} T x_k^{(j)} \chi_k^{(j)} \Big| L_1 \right\| \le \tau_p(T|\mathcal{H}(\mathbb{D}_1^n))\, n^{1/p-1/2} \times \left\| \left(\sum_{k=1}^{n} \left\| \sum_{j=1}^{2^{k-1}} x_k^{(j)} \chi_k^{(j)} \right\|^2 \right)^{1/2} \Big| L_\infty \right\|$$

  and hence by Theorem 5.1

$$\tau(T|\mathcal{H}(\mathbb{D}_1^n)) \le c\, \tau_p(T|\mathcal{H}(\mathbb{D}_1^n))\, n^{1/p-1/2}.$$

- In the case $p = 2$ the corresponding definition of weak Haar type 2 would obviously coincide with that of Haar type 2. That's why, we consider weak Haar type p only for $p < 2$.

- In Pisier's work [5] it was shown that a Banach space X is of Haar type p exactly if it admits an equivalent p-smooth renorming. A similar statement also holds for operators; see also [1] or the forthcoming book [4] for this connection.

- Pisier in [6] also introduced a concept of martingale type p, which again is equivalent to Haar type p. This follows from the considerations in [5].

We can now prove the main theorem of this article.

THEOREM 5.2: *If an operator $T \in \mathfrak{L}(X, Y)$ is of weak Haar type p for some $1 \leq p < 2$ then it is of Haar type r for all $r < p$.*

Proof: The proof follows essentially that of Sublemma 3.1 in Pisier [5]. The main new ingredient is the use of the results of Section 3.

Let $r < p$ and $(x_h^{(i)}) \subseteq X$. Set

$$S_r := \left\| \left(\sum_{k=1}^{n} \left\| \sum_{j=1}^{2^{k-1}} x_k^{(j)} \chi_k^{(j)} \right\|^r \right)^{1/r} \Big| L_\infty \right\|$$

and for $l = 1, 2, \ldots$ define

$$\mathbb{F}_l := \left\{ (k, j) \in \mathbb{D}_1^n \colon \frac{S_r}{2^{l/r}} < 2^{(k-1)/2} \|x_k^{(j)}\| \leq \frac{S_r}{2^{(l-1)/r}} \right\}.$$

Then it follows that

$$\sum_{\mathbb{D}_1^n} x_k^{(j)} \chi_k^{(j)} = \sum_{l=1}^{\infty} \sum_{\mathbb{F}_l} x_k^{(j)} \chi_k^{(j)}. \tag{5}$$

Moreover the sets $\mathbb{F}_l$ have local height less than 2^l. To see this last fact, choose $t \in [0, 1)$ and note that

$$\begin{aligned} S_r^r &\geq \sum_{\mathbb{D}_1^n} \|x_k^{(j)} \chi_k^{(j)}(t)\|^r \\ &\geq \sum_{\mathbb{F}_l \cap \mathbb{B}(t)} \|x_k^{(j)} \chi_k^{(j)}(t)\|^r > \sum_{\mathbb{F}_l \cap \mathbb{B}(t)} \left(\frac{S_r}{2^{l/r}} \right)^r = |\mathbb{F}_l \cap \mathbb{B}(t)|\, S_r^r\, 2^{-l}. \end{aligned}$$

This shows that

$$|\mathbb{F}_l \cap \mathbb{B}(t)| < 2^l. \tag{6}$$

By Theorem 3.8 and the definition of weak Haar type p, there exists a constant c such that

$$\tau(T|\mathcal{H}(\mathbb{F}_l)) \le \tau(T|\mathcal{H}(\mathbb{D}_1^{2^l})) \le c\,2^{l(1/p-1/2)}. \tag{7}$$

Since $r < p$ it follows that

$$\sum_{l=1}^{\infty} 2^{l(1/p-1/r)} = c_{pr} < \infty.$$

Hence (7) implies

$$\sum_{l=1}^{\infty} \tau(T|\mathcal{H}(\mathbb{F}_l))\,2^{l(1/2-1/r)} \le c\,c_{pr}. \tag{8}$$

Moreover we have

$$\begin{aligned} \sum_{\mathbb{F}_l} \|x_k^{(j)}\|^2 &= \sum_{\mathbb{F}_l} \int_0^1 \|x_k^{(j)}\chi_k^{(j)}(t)\|^2 dt = \int_0^1 \sum_{\mathbb{F}_l \cap \mathbb{B}(t)} \|x_k^{(j)}\chi_k^{(j)}(t)\|^2 dt \\ &= \int_0^1 \sum_{\mathbb{F}_l \cap \mathbb{B}(t)} 2^{k-1}\|x_k^{(j)}\|^2 dt \\ &\le \int_0^1 |\mathbb{F}_l \cap \mathbb{B}(t)| \left(\tfrac{S_r}{2^{(l-1)/r}}\right)^2 dt \le 2^{2/r} 2^{l(1-2/r)} S_r^2. \end{aligned} \tag{9}$$

Now (5), the definition of $\tau(T|\mathcal{H}(\mathbb{F}_l))$, (10), and (8) imply

$$\begin{aligned} \Big\| \sum_{\mathbb{D}_1^n} Tx_k^{(j)}\chi_k^{(j)} \Big| L_2 \Big\| &\le \sum_{l=1}^{\infty} \Big\| \sum_{\mathbb{F}_l} Tx_k^{(j)}\chi_k^{(j)} \Big| L_2 \Big\| \\ &\le \sum_{l=1}^{\infty} \tau(T|\mathcal{H}(\mathbb{F}_l)) \left(\sum_{\mathbb{F}_l} \|x_k^{(j)}\|^2 \right)^{1/2} \\ &\le 2^{1/r} \sum_{l=1}^{\infty} \tau(T|\mathcal{H}(\mathbb{F}_l)) 2^{l(1/2-1/r)} S_r \le 2^{1/r} c\, c_{pr}\, S_r. \end{aligned}$$

Finally Theorem 5.1 gives

$$\tau_r(T|\mathcal{H}(\mathbb{D}_1^n)) \le 2^{1/r}\, c\, c_{pr},$$

which completes the proof. ∎

Looking at this proof more closely and exploiting Corollary 4.2, we can even prove the following local variant of the previous result:

THEOREM 5.3: *If an operator $T \in \mathfrak{L}(X,Y)$ is of weak Haar type p for some $1 \le p < 2$ then there exists a constant $c \ge 0$ such that*

$$\tau_p(T|\mathcal{H}(\mathbb{D}_1^n)) \le c\,(1+\log n)$$

for all $n \in \mathbb{N}$.

Proof: Let $(x_k^{(j)}) \subseteq X$. As in the previous proof define

$$\mathbb{F}_l := \left\{ (k,j) \in \mathbb{D}_1^n \colon \frac{S_p}{2^{l/p}} < 2^{(k-1)/2}\|x_k^{(j)}\| \le \frac{S_p}{2^{(l-1)/p}} \right\}.$$

Again (5) and (6) hold. Let m be such that $2^m \le n < 2^{m+1}$. We now use Corollary 4.2 to construct modifications $\mathbb{F}'_1, \dots, \mathbb{F}'_{m+1}$ of the sets $\mathbb{F}_l$ of maximal cardinality. This is done inductively. If $|\mathbb{F}_1| \ge 3$ then we let $\mathbb{F}'_1 := \mathbb{F}_1$. Otherwise, by Corollary 4.2, there exist a subset $\mathbb{F}''_1 \subseteq \mathbb{D}_1^n \setminus \mathbb{F}_1$ of cardinality $3 - |\mathbb{F}_1|$ such that $\mathrm{lh}(\mathbb{F}_1 \cup \mathbb{F}''_1) \le 2$. Defining $\mathbb{F}'_1 := \mathbb{F}_1 \cup \mathbb{F}''_1$, we get that

$$\mathrm{lh}(\mathbb{F}'_1) \le 2^1 \qquad \text{and} \qquad |\mathbb{F}'_1| \ge 2^{2^1} - 1.$$

Now assume that $l \le m$ and that $\mathbb{F}'_1, \dots, \mathbb{F}'_{l-1}$ are already defined such that

$$\begin{array}{rlrl} \mathrm{lh}(\mathbb{F}'_{l-1}) & \le 2^{l-1}, & \sum_{k=1}^{l-1} |\mathbb{F}'_k| & \ge 2^{2^{l-1}} - 1, \\ \mathbb{F}'_{l-1} & \subseteq \mathbb{D}_1^n \setminus \bigcup_{k=1}^{l-2} \mathbb{F}_k, & \bigcup_{k=1}^{l-1} \mathbb{F}_k & \subseteq \bigcup_{k=1}^{l-1} \mathbb{F}'_k. \end{array}$$

If

$$\sum_{k=1}^{l-1} |\mathbb{F}'_k| + |\mathbb{F}_l| \ge 2^{2^l} - 1$$

then we let $\mathbb{F}'_l := \mathbb{F}_l$. Otherwise, by Corollary 4.2 (note that $2^l \le 2^m \le n$), there exists a subset $\mathbb{F}''_l \subseteq \mathbb{D}_1^n \setminus \mathbb{F}_l$ of cardinality $2^{2^l} - 1 - |\mathbb{F}_l|$ such that $\mathrm{lh}(\mathbb{F}_l \cup \mathbb{F}''_l) \le 2^l$. Defining

$$\mathbb{F}'_l := (\mathbb{F}_l \cup \mathbb{F}''_l) \setminus \bigcup_{k=1}^{l-1} \mathbb{F}'_k,$$

we get that $\mathrm{lh}(\mathbb{F}'_l) \le 2^l$, and

$$\sum_{k=1}^{l} |\mathbb{F}'_k| \ge \sum_{k=1}^{l-1} |\mathbb{F}'_k| + (|\mathbb{F}_l| + |\mathbb{F}''_l|) - \sum_{k=1}^{l-1} |\mathbb{F}'_k| = 2^{2^l} - 1.$$

Moreover, by construction

$$\mathbb{F}'_l \subseteq \mathbb{D}_1^n \setminus \bigcup_{k=1}^{l-1} \mathbb{F}_k \quad \text{and} \quad \bigcup_{k=1}^{l} \mathbb{F}_k \subseteq \bigcup_{k=1}^{l} \mathbb{F}'_k.$$

The last condition ensures that for $(k,j) \in \mathbb{F}'_l$ we have $(k,j) \notin \bigcup_{h=1}^{l-1} \mathbb{F}_h$ and hence

$$2^{(k-1)/2} \|x_k^{(j)}\| \leq \frac{S_p}{2^{(l-1)/p}}.$$

This construction yields a sequence $\mathbb{F}'_1, \ldots, \mathbb{F}'_m$ of length m. Finally, we let

$$\mathbb{F}'_{m+1} := \mathbb{D}_1^n \setminus \bigcup_{l=1}^{m} \mathbb{F}'_l.$$

Obviously $\mathrm{lh}(\mathbb{F}'_{m+1}) \leq n < 2^{m+1}$. Moreover, for $(k,j) \in \mathbb{F}'_{m+1}$,

$$2^{(k-1)/2} \|x_k^{(j)}\| \leq \frac{S_p}{2^{m/p}}.$$

Since again $\mathrm{lh}(\mathbb{F}'_l) < 2^l$ for $l = 1, \ldots, m+1$, we have (7) for $\mathbb{F}'_l$. Moreover

$$\sum_{l=1}^{m+1} \tau(T|\mathcal{H}(\mathbb{F}'_l))\, 2^{l(1/2-1/p)} \leq c\,(m+1) \leq c\,(1+\log n).$$

Furthermore, Inequality (10) holds with r replaced by p. Now, since

$$\bigcup_{l=1}^{m+1} \mathbb{F}'_l = \mathbb{D}_1^n,$$

we get

$$\begin{aligned}
\Big\| \sum_{\mathbb{D}_1^n} T x_k^{(j)} \chi_k^{(j)} \Big| L_2 \Big\| &\leq \sum_{l=1}^{m+1} \Big\| \sum_{\mathbb{F}'_l} T x_k^{(j)} \chi_k^{(j)} \Big| L_2 \Big\| \\
&\leq \sum_{l=1}^{m+1} \tau(T|\mathcal{H}(\mathbb{F}'_l)) \left(\sum_{\mathbb{F}'_l} \|x_k^{(j)}\|^2 \right)^{1/2} \\
&\leq 2^{1/p} \sum_{l=1}^{m+1} \tau(T|\mathcal{H}(\mathbb{F}'_l)) 2^{l(1/2-1/p)} S_p \\
&\leq 2^{1/p} c\,(1+\log n)\, S_p.
\end{aligned}$$

A glance at Theorem 5.1 completes the proof. ∎

6. Final remarks

To show that the notions of Haar type p and weak Haar type p do not coincide, we provide the following example.

Let $1 < p < 2$ and $\sigma_k := k^{-1/p'}$. Consider the diagonal operator D_s: $l_1 \to l_1$ associated with the sequence $s = (\sigma_k)$, which is given by

$$x = (\xi_k) \longmapsto D_s x = (\sigma_k \xi_k).$$

Then

$$\tau(D_s|\mathcal{H}(\mathbb{D}_1^n)) = \left(\sum_{k=1}^{n} k^{-2/p'}\right)^{1/2} \leq n^{1/p-1/2}$$

and hence D_s is of weak Haar type p. On the other hand

$$\tau_p(D_s|\mathcal{H}(\mathbb{D}_1^n)) = \left(\sum_{k=1}^{n} k^{-1}\right)^{1/p'} \geq \frac{1}{2}(1+\log n)^{1/p'}$$

and hence D_s is not of Haar type p. The tedious calculations are carried out in [8].

However, for *spaces* the situation seems to be more complicated.

Problem 1. Given $1 < p < 2$, does there exist a Banach space X that is of weak Haar type p but not of Haar type p?

In the Rademacher case, such examples are given by the well known modifications of the Tsirelson space (see Tzafriri [7]) which are even Banach lattices. It can also be shown that for Banach lattices the concepts of Rademacher and Haar type are the same. However, it seems to be unknown whether the same is true for the corresponding weak properties.

The examples above also suggest that a stronger version of Theorem 5.2 is true.

Problem 2. If T is of weak Haar type p, does there exist a constant $c \geq 0$ such that

$$\tau_p(T|\mathcal{H}(\mathbb{D}_1^n)) \leq c\,(1+\log n)^{1/p'}?$$

One more problem remains open. In the introduction it was mentioned that martingale type p and Haar type p are equivalent properties.

Problem 3. Is it true that weak martingale type p and weak Haar type p are the same?

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